PMM U.S.S.R., Vol.53, No.6, pp.828-830, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00
©1991 Pergamon Press plc

TWO FORMULATIONS OF AN ELASTO-PLASTIC PROBLEM*

F.M. ERLIKHMAN

Solutions of two elasto-plastic problems are compared when all the stress tensor components are continuous on the elasto-plastic boundary and when the tangential component undergoes a discontinuity.

Let us consider an infinite body with a circular hole of unit radius loaded by a constant force $\sigma_r = -p$, $\tau_{r\theta} = 0$. Let the stresses at infinity be determined by the Kolosov-Mushkelishvili potential (A_0, B_0, and B_2 are real numbers)

$$\Phi(z) = A_0, \quad \Psi(z) = B_0 + B_2 z^2$$

We assume that the Mises plasticity condition

$$(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2 = 4k^2$$

is satisfied in the plastic zone around the hole.

A discontinuity in the shear component of the stress tensor is generally possible on the unknown elasto-plastic boundary Γ. This circumstance results from non-linearity of the condition /1/ on the one hand, and also from the discontinuity of Poisson's ratio during passage through the elasto-plastic boundary /2/.

In the case of continuity of all the stress tensor components on the elasto-plastic boundary, we solve the problem by the method described in /3/. Let φ_1 and φ_2 be the "plastic" and "elastic" Airy functions. Let us introduce the function $\varphi_3 = \varphi_2 - \varphi_1$. It follows from the definition of the function φ_3 /3/

$$\frac{\partial^2\varphi_3}{\partial x^2} + \frac{\partial^2\varphi_3}{\partial y^2} = f(z), \quad \frac{\partial^2\varphi_3}{\partial x^2} - \frac{\partial^2\varphi_3}{\partial y^2} - 2i\frac{\partial^2\varphi_3}{\partial x\,\partial y} = B_0 + B_2 z^2 - 2k\exp(-2i\theta), \quad z \to \infty$$

$$f(z) = 4A_0 - 4k\ln z - 2k + 2p, \quad \theta = \arg z$$

By analogy with /3/ we introduce three analytic functions $\Phi_3(\eta)$ and $\Psi(\eta)$

$$\omega(\eta) = c\eta + c_2/\eta + c_2/\eta^2 + \ldots \tag{1}$$

We obtain the following boundary value problem to determine these functions:

$$4\,\mathrm{Re}\,\Phi_3(\eta) = \begin{cases} 0, & |\eta| = 1 \\ f(|c\eta|), & |\eta| \to \infty \end{cases} \tag{2}$$

$$2\,[(\overline{\omega(\eta)})/(\omega'(\eta))\,\Phi_3'(\eta) + \Psi_3(\eta)] = \begin{cases} 0, & |\eta| = 1 \\ 2B_0 + 2B_2(c^2\eta^2 + cc_1) - 2k\exp(-2i\theta_1), & |\eta| \to \infty \end{cases} \tag{3}$$

$$(\theta_1 = \arg\eta)$$

Solving the Dirichlet problem (2), we obtain

$$\Phi_3(\eta) = -k\ln\eta \tag{4}$$

It follows from the condition at infinity that $f(c) = 0$. As $\eta \to \infty$ we find because of $\omega(\eta) = c\eta$ /3/,

$$2(\overline{\omega(\eta)})/(\omega'(\eta))\,\Phi_3'(\eta) = -2k\exp(-2i\theta_1) \tag{5}$$

Comparing (5) and (3) we conclude that

$$\Psi_3(\eta) = B_0 + B_2(c^2\eta^2 + cc_1) + \Omega(\eta) \tag{6}$$

where $\Omega(\eta)$ is a regular function outside the circle $|\eta| = 1$.

Substituting (4) into (3), we obtain

$$\omega'(\eta)\,\Psi_3(\eta) = k\overline{\omega(\eta)}/\eta$$

Substituting (1) and (5) into this equality, we find

$$(c - c_1/\eta^2 - 2c_2/\eta^3 - \ldots)(B_2c^2\eta^2 + (B_2cc_1 + B_0) + \Omega(\eta)) = k(c/\eta^2 + c_1 + c_2\eta + \ldots), \quad c_i = \bar{c}_i \tag{7}$$

*Prikl. Matem. Mekhan., 53, 6, 1041-1044, 1989

We multiply the expression in the brackets and we equate coefficients of positive powers of η on the right and left-hand sides. The expressions on both sides of the equality (7) will here differ by a function that is regular outside $|\eta| = 1$ which is evidently allowable /3/. Consequently, we derive a system of equations to determine the coefficients in (1) and we conclude that the mapping function has the form

$$z = \omega(\eta) = c\eta + c_1/\eta + c_3/\eta^3, \quad c_1 = B_0 c/k, \quad c_3 = B_2 c^3/k \tag{8}$$

We now assume that the shear component of the stress tensor undergoes a discontinuity on Γ:

$$\sigma_t^p - \sigma_t^e = 4\sqrt{k^2 - \tau_n^2} \tag{9}$$

where σ_t^p, σ^e are the shear stresses from the plastic and elastic zones, respectively, (we note that if the one-dimensional problem $B_0 = B_2 = 0$ is considered, the plastic zone is greater in the discontinuous case). The conditions on the unknown boundary take the following form in the plane of the complex variable z:

$$\begin{gathered} 4\,\mathrm{Re}\,\Phi(z) = \sigma_n^p + \sigma_t^p - 4\sqrt{k^2 - \tau_n^2} \\ 2[\bar{z}\Phi'(z) + \Psi(z)] = (\sigma_y^e - \sigma_x^e + 2i\tau_{xy}^e) = (\sigma_t^e - \sigma_n + 2i\tau_n^e)\exp(-2i\gamma) \end{gathered} \tag{10}$$

where γ is the angle between the normal to Γ and the x-axis, and $\Phi(z)$ and $\Psi(z)$ are Kolosov-Muskhelishvili potentials. The quantity σ_t^e can be eliminated in the second formula of (10) by using (9). The stresses in the plastic zone are known

$$\sigma_r^p = 2k\ln r - p, \quad \sigma_\theta^p = 2k\ln r - p + 2k, \quad \tau_{r\theta} = 0 \tag{11}$$

On the unknown boundary Γ

$$\tau_n = {}^1/_2(\sigma_r - \sigma_\theta)\sin 2\beta = -k\sin 2\beta \tag{12}$$

where β is the angle between the normal to Γ and the radius-vector to Γ.

Substituting (11) and (12) into (10) and taking into account that $\beta - \gamma = -\alpha$, where α is the angle between the x axis and the radius-vector to Γ, we find

$$\begin{gathered} 4\mathrm{Re}\,\Phi(z) = 4k\ln r - 2p + 2k - 4k\cos 2\beta \\ 2[\bar{z}\Phi'(z) + \Psi(z)] = 2k\bar{z}/z - 4k\cos 2\beta\exp(-2i\gamma) \end{gathered}$$

We transfer to the parametric plane of the variable η by using the transformation (1). We then obtain the following boundary value problem to determine the three analytic functions

$$\begin{gathered} \varphi(\eta) = \Phi(\omega(\eta)), \quad \psi(\eta) = \Psi(\omega(\eta)) \text{ и } \omega(\eta): \\ 2[\varphi(\eta) + \overline{\varphi(\eta)}] = \begin{cases} H(\omega(\eta), \overline{\omega(\eta)}), & |\eta| = 1 \\ 4A_0, & |\eta| \to \infty \end{cases} \\ 2\left[\frac{\overline{\omega(\eta)}}{\omega'(\eta)}\varphi'(\eta) + \psi(\eta)\right] = \begin{cases} F(\omega(n), \overline{\omega(\eta)}), & |\eta| = 1 \\ 2B_2c^2\eta^2, & |\eta| \to \infty \end{cases} \\ H(\omega, \bar{\omega}) = 2k\ln(\omega\bar{\omega}) - 2p + 2k - 2k((a(\eta))/\overline{(a(\eta))} + \overline{(a(\eta))}/(a(\eta))) \\ F(\omega, \bar{\omega}) = -2k((\overline{\omega'(\eta)})^2\omega(\eta)\bar{\eta}^2)/((\omega'(\eta))^2\overline{\omega(\eta)}\eta^2)) \\ a(\eta) = (\omega'(\eta)\eta)/(\omega(\eta)) \end{gathered}$$

Let us examine the functional equation

$$2[(\overline{\omega(\eta)}/\omega'(\eta))\varphi'(\eta) + \psi(\eta)] = F(\omega(\eta), \bar{\omega}(1/\eta)) \tag{13}$$

For the right-hand side of this functional equation to be an analytic function outside the unit circle, it is necessary and sufficient /4/ to require pairwise agreement between the zeros of the functions $\overline{\omega(\eta)}$ and $\overline{\omega'(\eta)}$.

We will seek the mapping function in the form

$$\omega(\eta) = (\eta^2 - b^2)^n c_{2n-1}/\eta^{2n-1}$$

Substituting this expression into (13) and expanding all the functions in series in the neighbourhood of the infinitely remote points, it can be noted that $n = 2$. Therefore, the mapping function has the form

$$\omega(\eta) = c_3/\eta - 2b^2c_3/\eta + b^4c_3/\eta^3 \tag{14}$$

and because of the symmetry of the problem b and c_3 are real numbers. (For the elastic domain to be bounded everywhere with the plastic domain, it is necessary that the circle of unit radius, that is the hole contour, be within an ellipse; this results in the condition $c_3(1 - 2b^2 + b^4) \geqslant 1$). Substituting the specific form of the mapping function (14) into the right-hand side of the functional Eq. (13), we find

$$F(\omega, \bar{\omega}) = -2k(1 + 3b^2\eta^2)^2\eta^2/(\eta^2 + 3b^2)^2, \quad |b| < 1/\sqrt{3}$$

Since, the variable η occurs in $F(\omega, \bar{\omega})$ in an even power throughout, we have

$$F = -18kb^4\eta^2 + \alpha_0 + \alpha_{-2}/\eta^2 + \ldots \tag{15}$$

The left-hand side of (13) is expanded in the form

$$2B_2c_3^2\eta^2 + 2\gamma_0 + \gamma_{-2}/\eta^2 + \ldots \tag{16}$$

(the symmetry of the problem is used). Equating coefficients of η^2 in (15) and (16), we obtain

$$-18kb^4 = 2B_2c_3^2$$

It is possible to proceed further as follows. We have for the harmonic function $\operatorname{Re}\varphi(\eta)$ by the mean-value theorem

$$\frac{1}{2\pi}\int_0^{2\pi} H(\omega, \bar{\omega})\, d\theta = 4A_0 \tag{17}$$

Let the constants B_0, B_2 and c_3 be known such that the constant b found from (17) satisfies the condition $|b| < 1/\sqrt{3}$. In this case (18) can be utilized to determine A_0. Comparing the free terms in (13) we obtain a relationship connecting A_0, B_0 and B_2, that can be considered the condition for the problem to be solvable for given B_0, B_2 and c_3:

$$2\gamma_0 = \alpha_0 \tag{18}$$

Here

$$\gamma_0 = b^4a_3' + (B_0 - 4B_2b^2c_3^2), \quad \alpha_0 = -12kb^2$$

$$a_3' = -2a_2, \quad a_2 = \frac{1}{2\pi i}\int_{|\tau|=1} H(\omega(\tau), \overline{\omega(\tau)})\, \tau\, d\tau$$

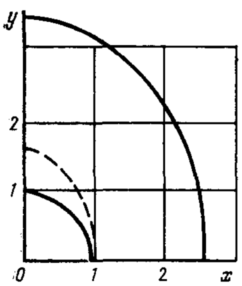
It follows from (19) that

$$B_0 = 3B_2b^2c_3^2 + 2b^4a_2 - 6kb^2 \tag{19}$$

We will carry out a parametric investigation of solutions (8) and (14). The solution is meaningful when $|\omega(t)| \geqslant 1, |t| = 1$ and the function $\omega'(\eta)$ has no zeros for $|\eta| \geqslant 1$. In the "continuous" case, these conditions impose the following constraints on the domain of parameter variation:

$$c(1 - B_0/k + B_2c^2/k) \geqslant 1, \quad (1 - B_0/k - 3B_2c^2/k) > 0 \tag{20}$$

$$c = \exp((4A_0 - 2p + 2k)/(4k))$$



In the "discontinuous" problem, the corresponding constraints have the form

$$c_3(1 - 2b^2 + b^4) \geqslant 1, \quad |b| < 1/\sqrt{3}, \quad b^4 = -B_2c_3^2/(9k) \tag{21}$$

This last relationship (21) shows that the condition $B_2 \leqslant 0$ should be satisfied. In particular, if $B_2 = 0$ and therefore, $b = 0$ also, then it follows from condition (19) that $B_0 = 0$ also. Comparing the solutions (8) and (14) we see that the elasto-plastic boundaries are similar only in the one-dimensional problem $(B_0 = B_2 = 0)$, which are circles of different radii.

It should be noted that the elasto-plastic problem can also be solved in the continuous case by the method of functional equations /4/ utilized to solve the problem in the discontinuous formulation.

The positions of the elasto-plastic boundary in the continuous (dashed line) and discontinuous (solid line) cases are shown in the figure for $A_0 = 0.632$, $B_0 = -0.23$, $B_2 = -0.05$, $k = 1$, $p = 0.2$. Initially B_2 and $c_3 = 3$ were given here. Then A_0 was determined from condition (18) and further B_0 also from (19).

REFERENCES

1. KHRISTIANOVICH S.A., MIKHLIN S.G. and DEVIDSON B.B., Some Problems of the Mechanics of a Continuous Medium, Izd. Akad. Nauk SSSR, Moscow, 1938.
2. IVLEV D.D., On determining the displacements in the L.A. Galin problem, PMM, 21, 5, 1957.
3. GALIN L.A., Elasto-Plastic Problems, Nauka, Moscow, 1984.
4. ANIN B.D. and CHEREPANOV G.P., The Elasto-Plastic Problem, Nauka, Novosibirsk, 1983.

Translated by M.D.F.